Recall the following stuff in class.

DEFN 2. Let f be measurable and $f \geq 0$. We define, for a measure space $(X, \mathcal{M}, \mu)$,

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple and} \\ 0 \leq s \leq f \end{array} \right\}.$$

This is the defn we used in class.

DEFN 2'. For a measure space $(X, \mathcal{M}, \mu)$, let f be measurable and $f \geq 0$.

we define

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple,} \\ 0 \leq s \leq f \\ \mu(\{x \mid s(x) > 0\}) < +\infty \end{array} \right\}$$

I mentioned the following fact in class

FACT: If $(X, \mathcal{M}, \mu)$ is σ-finite, then DEFN 2 and DEFN 2' are equivalent.

Remark: This fact can be proved directly from defns.

We will need this fact.

What I will prove is this:

extra requirement added

Prop: Let $(X, \mathcal{M}, \mu)$ be a measure space which is σ-finite. Let f and g be two measurable positive functions. Then $\int_X f+g \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.

Proof:

We already proved $\int_X f+g \, d\mu \geq \int_X f \, d\mu + \int_X g \, d\mu$ in class.

Just need to show that $\int_X f+g \, d\mu \leq \int_X f \, d\mu + \int_X g \, d\mu$. (*)

We will mention the following claim first.

Claim 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and g be two positive measurable functions. If $f \leq g$, then $\int_X f \, d\mu \leq \int_X g \, d\mu$.

Proof: Can be proved directly from the fact that "If s is simple and $0 \leq s \leq f$, then $0 \leq s \leq g$".

Claim 2. Let $(X, \mathcal{M}, \mu)$ be a measure space w/ $\mu(X) < +\infty$. Let f and g be two measurable functions s.t $|f| \leq M$ and $|g| \leq M$ for certain $M > 0$. Then

$$\int_X f+g \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.$$

Proof: Just ~~need~~ note that we can find simple functions $\{f_n\}$ and $\{g_n\}$ such that $f_n \to f$ uniformly and $g_n \to g$ uniformly (this is possible because $|f| \leq M$ and $|g| \leq M$). Also note that $\mu(X) < +\infty$, easy to check that the uniformly convergence of f_n to f ensures that

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

Similarly, $\int_X g \, d\mu = \lim_{n \to \infty} \int_X g_n \, d\mu$

3

Note that $f_n + g_n \to f+g$ uniformly. it follows that

$$\lim_{n\to\infty} \int_X f_n + g_n \, d\mu = \int_X f + g \, d\mu.$$

As each f_n and g_n is simple. we have

$$\int_X f_n + g_n \, d\mu = \int_X f_n \, d\mu + \int_X g_n \, d\mu \quad \forall n \in \mathbb{N}_{\geq}.$$

Let $n \to \infty$, and we are done. $\boxed{\checkmark}$

Now, we are in position to prove $(*)$ in page 2).

Review the set up : f, g are measurable, positive. (X, m, μ) is σ-finite.

Thus DEFN 2 is equivalent to DEFN 2'. That is. while defining $\int_X f \, d\mu$ and $\int_X g \, d\mu$. we just need to consider those positive simple functions that is $\leq f$ (or $\leq g$) and is non-zero on a subset w/ finite measure.

We need to prove that $\int_X f+g \, d\mu \leq \int_X f \, d\mu + \int_X g \, d\mu$.

Just need to show that $\forall$ simple function s with $0 \leq s \leq f+g$. and $\mu(\{x \mid s(x) > 0\}) < \infty$, we have

$$\int_X s \, d\mu \leq \int_X f \, d\mu + \int_X g \, d\mu.$$

Use E to denote $\{x \mid s(x) > 0\}$.

Let $h_1 = \min(f, s)$ and $h_2 = s - h_1$.

Claim: $h_1 \geq 0$, $h_2 \geq 0$, $h_1 \leq s$, $h_2 \leq s$. (This is obvious).

Claim: $h_1\big|_{X-E} \equiv 0$, $h_2\big|_{X-E} \equiv 0$ (Follows from the previous claim)

Claim: $h_1 \leq f$, $h_2 \leq g$.

Proof: $h_1 \leq f$ is just trivial., We will show that $h_2 \leq g$.

$\forall x \in X$. we consider the following two cases

Case 1. $s(x) \leq f(x)$

In this case, $h_1(x) = s(x)$. $h_2(x) = s(x) - h_1(x) = 0 \leq g(x)$.

Case 2 $s(x) > f(x)$.

In this case, $h_1(x) = f(x)$. As $s(x) = h_1(x) + h_2(x) \leq f(x) + g(x)$, it follows that $h_2(x) \leq g(x)$. $\square$.

Now, back to the proof:

$$\int_X s \, d\mu = \int_X \chi_E \cdot s \, d\mu$$

$$= \int_E s \, d\mu$$

$s = h_1 + h_2$, $s\big|_{X-E} = 0$.

$$\overset{h_i|_{X-E} = 0,\ i=1,2}{=\!=} \int_E h_1 + h_2 \, d\mu$$

• $\mu(E) < \infty$

• h_1, h_2 are bounded b/c s is bounded and using claim 2 on page 2.

$$= \int_E h_1 \, d\mu + \int_E h_2 \, d\mu$$

• Can check from defn.

$$= \int_X \chi_E \cdot h_1 \, d\mu + \int_X \chi_E \cdot h_2 \, d\mu$$

5

$\cdot\ h_1 \le f,\ h_2 \le g$

$\cdot$ claim 1 on page 2.

$$\le \int_X f\,d\mu + \int_X g\,d\mu .$$

Then we are done. $\blacksquare$